Classical nonlinearity and quantum decay: The effect of classical phase-space structures

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We investigate the decay process from a time dependent potential well in the semiclassical regime. The classical dynamics is chaotic and the decay rate shows an irregular behavior as a function of the system parameters. By studying the weak-chaos regime we are able to connect the decay irregularities to the presence of nonlinear resonances in the classical phase space. A quantitative analytical prediction that accounts for the numerical results is obtained.

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I. INTRODUCTION

In the past years, many conjectures have been put forward, and tested in various system models, in order to answer the fundamental question in quantum chaos: what is the signature of classical chaos in the quantum world? Among these, one of the most intriguing is the idea that classical chaos can induce large-scale fluctuations on a genuine quantum phenomenon such as the tunneling process. Starting from the seminal paper of Davis and Heller [1], who first noted the occurrence of coherent tunneling between regular tori separated by a chaotic region, the influence of classical chaos on quantum tunneling has been verified in many systems and is now accepted in the literature as a fingerprint of classical nonintegrability. It is very simple to describe this effect. Let us consider a system that is classically chaotic and invariant under a symmetry operation, for example, space inversion. If the classical system supports a regular torus, by symmetry there might also be a second torus that is distinct from its symmetric partner, for instance, two symmetric tori encircling the bottom of the two wells of a double-well potential. Moreover, let us suppose that the two tori are large enough to support quantum states. Under this condition, the quantum system will show coherent tunneling between the states located in the two symmetric tori. If now one system parameter is changed (e.g., $\hbar$), contrary to the expectations of ordinary semiclassical analysis, the tunneling rate shows strong irregularities that can increase or decrease the rate by orders of magnitude.

The tunneling fluctuation is usually interpreted in terms of a process known as ''chaos assisted tunneling'' (CAT) [2–10]. An intuitive view of the CAT process could be as follows. The presence of regular and stochastic motion in the classical phase space corresponds, from a quantum point of view, to the possibility of having two kinds of states: regular ones localized inside the symmetric tori and chaotic states which, being extended through the chaotic region, display a non-negligible overlap with regular regions. The fluctuations in the tunneling rate are thus explained in terms of a three-state tunneling process. The quantum particle first tunnels from the localized state to an extended chaotic one and then from this to the state located in the symmetric torus.

These features motivated the widespread idea that classical chaotic trajectories can have an active role in the quantum process, helping or ''assisting'' the quantum particle to tunnel between the symmetric tori. Along this line, path integral techniques have been used to calculate the contribution to the tunneling stemming from complex orbits that connect the symmetric regular tori through the classical stochastic layer [7].

However, a real quantitative theory of CAT is still lacking. The main reason for this can probably be found in the chaotic nature of the third state that prevents simple analytical treatments. Moreover, there are some aspects of the phenomenon that do not seem to fit properly the intuitive interpretation given by the CAT picture. For example, the presence of strong decreases in the tunneling rate which, together with the enhancements, occur as a result of a parameter change, contradicts the idea of a tunnel process being ''assisted'' by chaos. Another controversial aspect is related to whether chaos is essential for this phenomenon, being possible to find similar behaviors in nonlinear system that are not classically chaotic. We shall further discuss these issues in the next section.

The purpose of this paper is to assess whether this picture applies also to a different tunneling process, namely, to the quantum escape of a particle that has been initially located inside a potential well. From a classical point of view, it is clear that the particle can overcome the potential barrier of the well only if its energy is larger than the barrier height, while in the quantum framework the tunneling across the classically forbidden region is always present. Clearly, this situation is modified when, including the ingredient of chaos, we perturb the system by adding a forcing term, i.e., a time dependent external force. The perturbation disturbs the regular motion of the classical particle and, by increasing its energy, makes it possible for the particle to escape over the barrier. In the meanwhile, also the quantum process of tunneling changes due to the modification of the potential and both the processes contribute to the decay [11]. Our purpose is to choose a region of the system parameters where the classical and the quantum contribution to the decay can be separated, in order to study the properties of the latter pro-

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cess in connection with the chaotic features of the classical phase space.

Our plan to study a decay process will lead us to deal with an unbounded system, with a continuous spectrum, and this will prevent us from using standard methods, like the diagonalization of the Floquet dynamic operator [12], to obtain directly the level splittings responsible for the tunneling. Anyway, we shall be able to analyze the system by resorting to a somewhat simpler method: we shall calculate numerically the time evolution of a quantum state initially located in the potential well and, by studying the decay of the population in the well, we shall be able to obtain the relative strength of the tunneling as a function of the system parameters. This will allow us to point out the differences between this process of chaos-assisted decay and the tunneling processes in the presence of chaos (CAT).

Finally, we shall be able to show that the picture that singles out the classical nonlinear resonances as the main factor responsible for the fluctuations of the tunneling rate [9], applies also to this context. We shall review the semiclassical prediction of Ref. [9], which is valid for bounded systems, in Sec. V, and we shall verify its validity for decay processes.

The following section will be devoted to review the quantum levels dynamics at the basis of CAT, to better understand the similarities and the differences between this phenomenon and the perturbation of quantum decay that is the subject of this paper.

II. AVOIDED CROSSINGS AND TUNNELING IRREGULARITIES

As we discussed in the previous section, the CAT is seen as the result of the interaction between regular and chaotic states in systems that are classically chaotic. This interpretation is confirmed by the level dynamics of the tunneling system. A typical situation is sketched in Fig. 1 where we can observe the change of two quasidegenerate levels, which correspond to the pair of tunneling regular states, as a system parameter is varied. In almost the entire parameter range the splitting between the two states, and so the direct tunneling probability, changes smoothly. However, it may occur that, once the parameter is changed, a third level (dashed line) crosses the two quasidegenerate levels. In the generic case, states belonging to the same symmetry class do not cross each other, therefore, the appearance of a third *colliding state* gives rise to *avoided crossing* with the state of the doublet that belongs to the same symmetry class. The avoided crossing has a twofold consequence on the tunneling process under study. First, in the vicinity of the crossing we cannot consider the tunneling as a process involving only the two quasidegenerate states. Under this condition, the standard two-state tunneling becomes a resonant three-state process. Second, since the colliding third state modifies the energy level of only one of the doublet states, the splitting of the two levels changes. The level modification is, loosely speaking, *proportional* to the system nonlinearity but, given the small value of the tunneling splitting in the semiclassical regime, the avoided crossing can produce a dramatic modification of

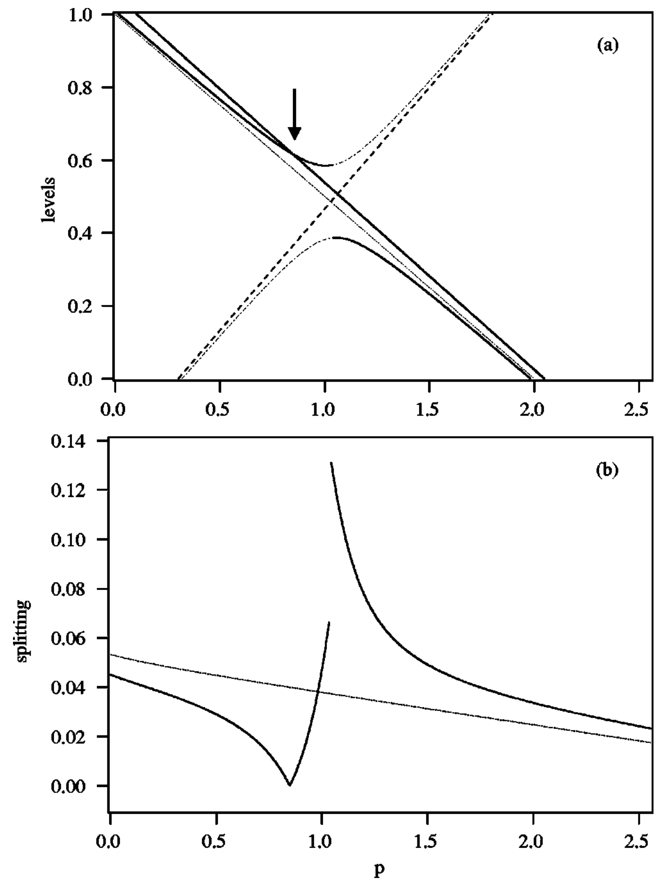

FIG. 1. Sketch of the typical behavior of the energy levels of a classically chaotic quantum system as the parameter p is changed. In (a) the two solid thick lines describe a couple of quasidegenerate levels of different symmetry. The thin dotted line represents the lower level of the tunneling doublet in the unperturbed case. The dashed line describes a colliding third level. In (b) we show the splitting of the two quasidegenerate levels in the perturbed and unperturbed case (thin dotted line). All the units are arbitrary.

the tunneling rate, even in the case of very weak chaos. It is important to point out that the rate can increase by several orders of magnitude as well as vanish (see the arrow in Fig. 1) according to the value of the parameters. Moreover, due to the fact that the energy spectrum, in the nonlinear case, does not show any regularity, the crossings with a third level do not follow a regular pattern, and the overall behavior of the tunneling rate appears to be an irregular sequence of peaks [4–7] instead of the smooth behavior expected in the regular systems.

As anticipated, a controversial aspect of this effect is the nature of the third state that crosses the tunneling doublet. The key point of the CAT picture is that the perturbation in the energy splitting is relevant only for those crossings involving colliding third states that are located in the chaotic region. However, it is well known that avoided crossings can be found in completely regular systems too, this being, in fact, the counterpart of the existence of classical nonlinear systems that are not chaotic.

The connection between tunneling and avoided crossings has been extensively studied in the last 20 years [13–19]: in regular systems, as a connection between classical behavior and Fermi resonance [18,19], or in classically chaotic systems, to look for a definition of quantum chaoticity along the lines of the Chirikov criterion for the birth of classical chaos [15,17].

In those papers, the attention was focused on a two-state phenomenon, namely, on the mixing of the two states that avoid each other, mixing that can be interpreted as a tunneling process if the two quantum states are located in classically separated region of the space, or, more generally, of the phase space. Moreover a connection was assessed between this phenomenon and the existence of nonlinear classical resonances [14–16]. More precisely, as stated by Roberts and Jaffé, ‘‘if two quantum mechanical zero-order states exhibit a *n:m* Fermi resonance [and thus an avoided crossing], then the classical dynamics associated with the matrix element connecting the two states should exhibit a *n:m* nonlinear resonance.’’

In our opinion this is a good starting point for a theoretical investigation of CAT. The tunneling irregularities are in fact connected to avoided crossings, as shown in Fig. 1, while the nonlinear classical resonances are the primitive structure at the basis of classical chaoticity. We thus believe that CAT can be seen as the effect of the superposition of several avoided crossings. A process similar to the birth of classical chaos due to the superposition of isolated nonlinear resonances. A path to quantum chaos that had already been discussed, even if without referring to tunneling dynamics [15,17].

Nevertheless, due to the fact that nonlinear resonances are present in systems that are not chaotic, it is clear that chaos is not a required ingredient for tunneling irregularities. For example, by studying the connection between tunneling irregularities and nonlinear resonances in a simple one-dimensional driven system, the authors of Ref. [9] showed that strong tunneling fluctuations are present also in the almost integrable case, when the third state responsible for the fluctuation is by no means chaotic.

III. THE MODEL: CLASSICAL DYNAMICS

In order to analyze the influence of classical chaos on the quantum process of escape from a potential well, we introduce a simple one-dimensional forced system described by the following Hamiltonian:

$$H=\frac{p^2}{2}+V(q,t), \tag{1}$$

$$V(q,t)=V_0[1-\cos(2q)]+\epsilon[1-\cos(2q-\Omega t)],$$

$$q=[-\pi,\pi], \quad V(q,t)=0 \ \text{ otherwise.} \tag{2}$$

The particle is located initially inside a potential well that has the form of a sinusoidal function extended over two periods, as shown in Fig. 2, and it is forced by a time periodic perturbation that is considered to be small compared to the static potential, i.e., $\epsilon \ll V_0$ [20].

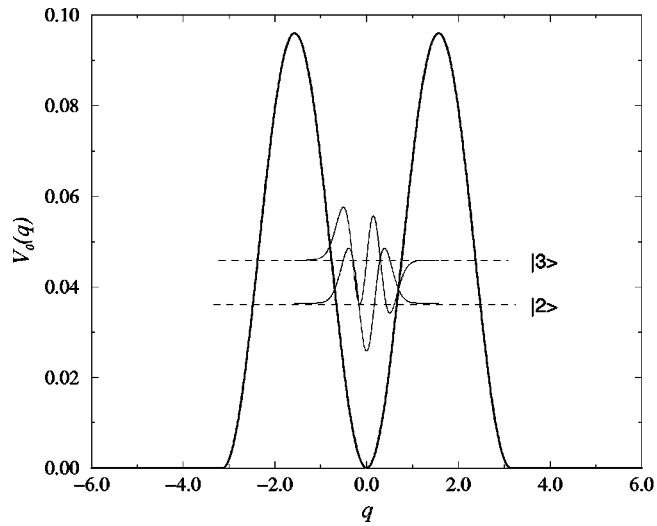

FIG. 2. The unperturbed potential of Eq. (2). We also show the wave function of the two states used as initial condition in the numerical calculations. The values of the parameters are $V_0 = 0.048$ and $\hbar = 0.025$. All the units are arbitrary.

We chose a perturbation term that turns system (1) into a *double-resonance*-like Hamiltonian. This choice is dictated by the need for simplicity. It is indeed clear that, as long as we limit our analysis to small perturbations, the particular form of the external forcing does not affect the generic features of the decay process we want to study. On the other hand, the adoption of Hamiltonian (1) presents many benefits. All the relevant information concerning the dynamic properties of our model can be derived from the dynamics of a well-known system, the *double-resonance* Hamiltonian, which corresponds to Eq. (1) with periodical boundary conditions [21,22].

The presence of a periodic perturbation in Eq. (1) breaks the integrability of the classical Hamiltonian. The most important features of this condition is the appearance of nonlinear resonances in the phase space together with regions characterized by extended chaotic motion (stochastic layer). The relevance of the chaotic motion depends on the strength ϵ of the perturbation term, so that the system can be more or less chaotic. In Fig. 3 we show a stroboscopic mapping of the dynamics, namely, the position in phase space at fixed intervals of time that are integer multiples of the forcing term period $T=2\pi/\Omega$, for a generic weak-chaos case. Some nonlinear resonances and the stochastic layer around the separatrix are clearly visible.

The nonlinear resonances are the visible consequences of the small denominators problem. These are related to the secular terms that appear in the perturbative solution of the equation of motion of nonintegrable systems. In the weak-chaos condition their position in the phase space can be obtained by considering the effect of the time-dependent term as a perturbation on the dynamics expressed by the constant Hamiltonian

$$H_0=\frac{p^2}{2}+V_0[1-\cos(2q)]\big|_{-\pi<q<\pi}. \tag{3}$$

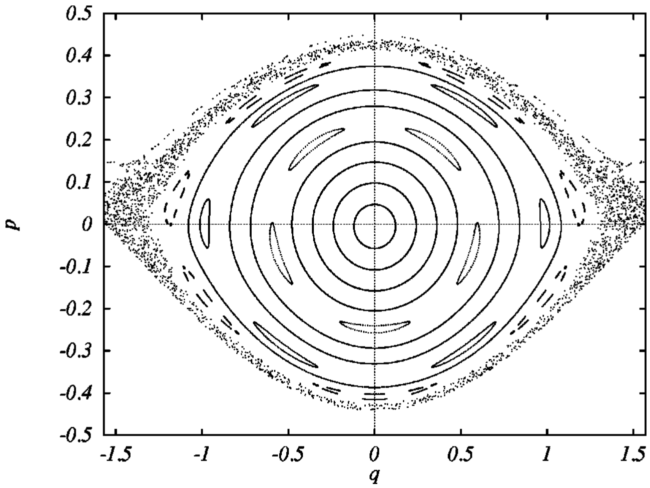

FIG. 3. The classical dynamics inside the well. Stroboscopic Poincaré map. The values of the parameters are $V_0=0.048$, $\epsilon=0.005$, $\Omega=2$. All the units are arbitrary.

The Kolmogorov-Arnol'd-Moser (KAM) theorem [23] states that, as long as the perturbing term can be considered *small*, the main part of the phase space remains practically unperturbed and that only the tori that are resonant with the forcing term are destroyed and replaced by chains of islands like the ones shown in Fig. 3. The resonant condition can be written as

$$\omega_0(E)=\frac{m}{n}\Omega, \tag{4}$$

where n and m are integer numbers and $\omega_0(E)$ is the frequency of the unperturbed motion inside the well that depends on the energy E. In our case, it is possible to express $\omega_0(E)$ in terms of the elliptic function $K(k)$ as

$$\omega_0(E)=\pi\sqrt{V_0}/K(k),$$
$$k\equiv(E+V_0)/2V_0. \tag{5}$$

The result of Eq. (4) would actually indicate that all the tori are destroyed by the perturbation, being the rational numbers dense among the real ones. Nevertheless, the KAM theorem assures that the effects of the perturbation becomes smaller and smaller with increasing the order of the resonance, i.e., with increasing the numerator m. This is clearly visible in Fig. 3 where we can only recognize the chains corresponding to $m=1$, i.e., the 1/5, 1/6, and 1/7 resonances. However, even for small perturbation, in the neighborhood of the separatrix of the unperturbed system the motion is always dominated by chaotic dynamics. In other words, trajectories that in the absence of perturbation are bounded inside the potential well, can now overcome the energy barrier and eventually escape from the well. Under the generic weak-chaos condition, the stochastic layer around the separatrix is dynamically separated from the phase-space region corresponding to bounded trajectories by unbroken tori, so that the process of escape driven by chaos is limited in phase space. Therefore, in the classical case, the decay of the population of the well is possible only if particles are initially placed inside the stochastic region. This process has been extensively studied in the last years in various classical and quantum models; the most known is probably the hydrogen atom in the presence of a strong radiation field [24,25]. In that system the dynamic process described above leads to the ionization of the atom. However, in this paper we want to focus on the connection between chaos and processes that would be classically impossible, such as quantum tunneling. For that reason we shall analyze the decay of the well population for a quantum particle initially located in the phase space region corresponding, even in the presence of chaos, to bounded motion. In connection with this, it is worthwhile to remember that in quantum mechanics the situation is never simple. Whatever the initial condition is, the wave function cannot be sharply located inside a finite region but exhibits smooth decreasing tails that extend over the stochastic layer. Therefore, to keep the classical chaotic diffusion process as small as possible, we shall consider a weak-chaos regime with a small stochastic layer such as the one shown in Fig. 3, and in addition to this, we shall study the decay of quantum states deeply localized inside the potential well.

IV. THE MODEL: QUANTUM DYNAMICS

We studied the quantum decay from the well of Fig. 2 by integrating numerically the time dependent Schrödinger equation associated with Hamiltonian (1). This can be done by using a FFT splitting algorithm [26] and absorbing boundary conditions [27] as described, for example, in Ref. [11].

In order to single out the effect of the chaotic perturbation on the process of quantum decay, it is necessary to choose as initial condition a state localized inside the well with an unperturbed dynamics as simple as possible. The eigenstates of Hamiltonian (1) with $\epsilon=0$ do not seem to be a proper choice in this context with system (1) being an open system (continuous spectrum with stationary states that do not have finite support inside the well). We thus resorted to use as initial condition metastable states that have a negligible *internal dynamics* and a long enough unperturbed lifetime inside the well. These are the *resonances* of the potential well of Fig. 2 defined in the quantum theory of scattering.

It is worthwhile to make the following remark. Since, in the explored parameter range, the decay probability is negligible, the adoption of the eigenstates of Hamiltonian (3) supplemented by periodical boundary conditions would lead essentially to the same results. As a first approximation, we can thus consider the initial states as stationary state of the unperturbed system. For the sake of simplicity, in the following we shall refer to them as "eigenstates" of the unperturbed Hamiltonian.

In Fig. 4 we show the time evolution of the population inside the well, $P(t)=\int_{-\pi/2}^{\pi/2}|\psi(q,t)|^2dq$, for different values of the forcing frequency Ω. This figure has been obtained by choosing as initial condition the fourth eigenstate $|3\rangle$ of the unperturbed well. It can easily be realized that the decay probability can be strongly enhanced by the forcing term even in the small perturbation regime ($\epsilon=0.005$). Note that

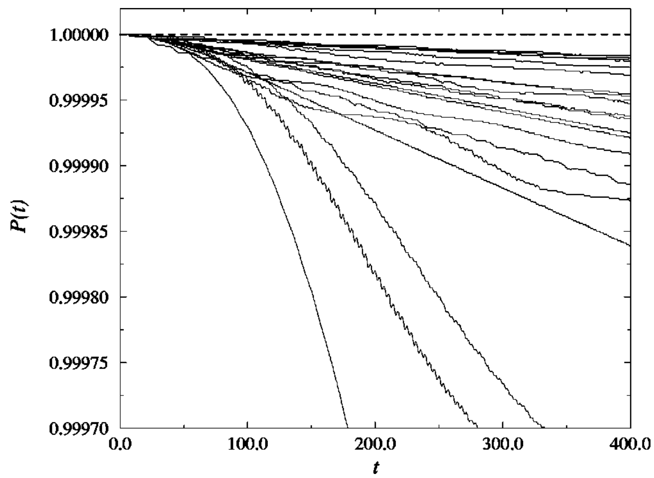

FIG. 4. The time evolution of the well population in arbitrary units. The different curves correspond to 20 different values of Ω included between $\Omega=1.5$ and $\Omega=2.5$. The values of the parameters are $V_0=0.048$, $\epsilon=0.005$, $\hbar=0.025$, and the initial state is the fourth eigenstate of the unperturbed well. The horizontal dashed line corresponds to the unperturbed $\epsilon=0$ case.

in the unperturbed case the population decay is not visible in the scale of this figure, the population remaining practically unchanged in the studied interval of time. This shows that the time scale of the unperturbed process is much longer than the maximum time that we explored numerically and also confirms that, within the observed time, the chosen initial states can indeed be regarded as ''stationary states'' of the unperturbed system.

To obtain a more quantitative representation of the phenomenon we could define the decay rate as the inverse of the time integral of $P(t)$, namely, the inverse of the area contained under the curves of Fig. 4. This would imply a very long numerical simulation, up to a time where the population has completely leaked out. However, we are not interested in the absolute magnitude of the decay rate, but only in its relative strength as a function of the system parameters. Thus, we can simply calculate the time integral of $P(t)$ up to a certain time t_{max} and measure the rate by studying the quantity

$$R=1-\frac{1}{t_{max}}\int_0^{t_{max}}P(t)dt \qquad (6)$$

as a function of the frequency Ω of the forcing term. Clearly, the values of R depend on t_{max}, but this dependence does not qualitatively affect the results, if the integration time t_{max} is large enough.

As shown in Fig. 5, R shows a sequence of peaks, similar to a resonant dependence on the frequency of the perturbation. We repeated the calculation for two different initial conditions, by choosing the third and the fourth eigenstates of the unperturbed well. The decay rate for the third state is smaller, as expected, since this state is more deeply located in the potential well, but in both cases we found a similar behavior even if the position of the peaks and their intensity are not the same. Except for the presence of the peaks, the

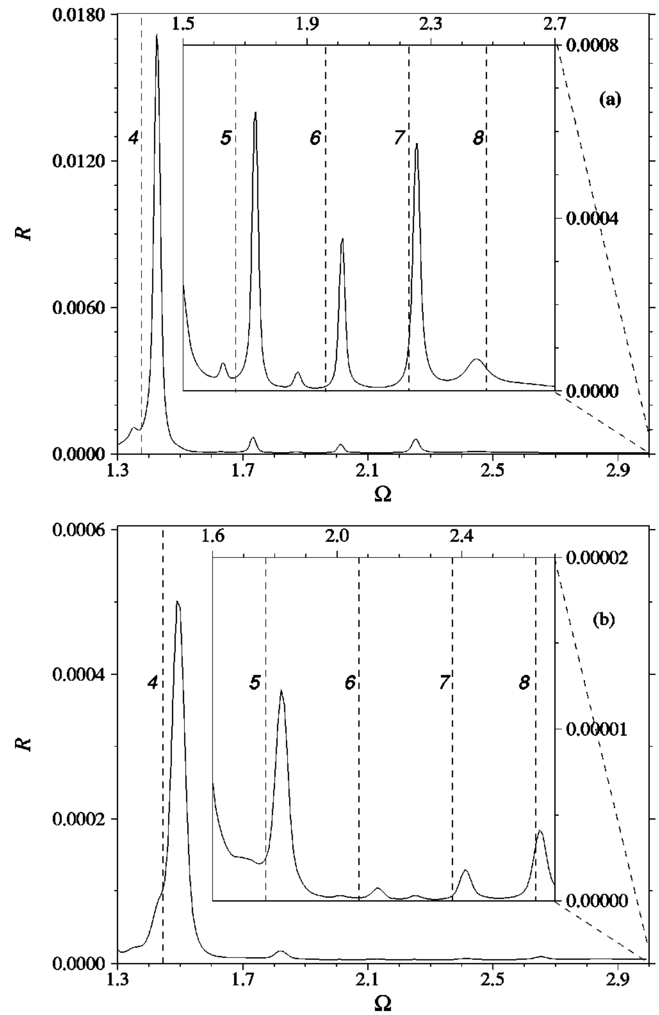

FIG. 5. R, see Eq. (6), calculated with $t_{max}=400$, as a function of the driving frequency Ω. The values of the parameters are $V_0=0.048$, $\epsilon=0.005$, $\hbar=0.025$. The two figures correspond to different initial conditions: (a) is obtained by choosing as initial state the fourth eigenstate of the unperturbed well, (b) by choosing the third. The numbers and the vertical dashed lines refer to the classical nonlinear resonances as discussed in Sec. V. The insets contain enlarged views. Arbitrary units.

figures show that, as a general tendency, the decay increases as the forcing frequency decreases. This can be understood using arguments based on the classical dynamics of system (1). Indeed, as shown in Fig. 7, the chaotic features of the classical phase space (i.e., width of the stochastic layer and of the nonlinear resonances) increase by reducing Ω, thus demonstrating that the forcing term becomes more important when Ω gets smaller. Eventually, for extremely small Ω, the stochastic layer becomes so wide that, for the chosen initial conditions, the escape from the well via the direct classical process becomes the dominant process. Since we are rather interested in the quantum mechanism of escape from the well, we shall not explore the condition of small Ω's corresponding to strong classical chaos.

In addition to this, there is a further reason to limit the analysis to not too small Ω's. Our evaluation of the decay rate, as the integral of the surviving population in the well over a finite time, is meaningful only if the time of integration is much longer than the period of the perturbation, and, therefore, we shall limit ourselves to study the range of large Ω $(2\pi/\Omega \ll t_{max})$.

However, it is important to point out that the numerical results represented in Figs. 4 and 5 do not account for the process of classical escape from the well, even for the smallest used value of Ω. A direct calculation demonstrates indeed that the classical decay is always negligible in the considered parameter range of Ω. This can easily be assessed by numerically integrating system (1) using a classical initial particle distribution mimicking the phase-space representation of the initial quantum state (see Sec. VI for details). Even for the smallest value of Ω used to obtain the results of Figs. 4 and 5, the classical population in the well does almost not change with time.

Therefore, the peaks and the general tendency of Figs. 4 and 5 are genuine quantum effects, but, while the latter can be associated with the increased effectiveness of the forcing term in Eq. (1) with decreasing Ω, the former effect does not have any simple interpretation. The next section will be devoted to deepening the understanding of this issue.

V. A SEMICLASSICAL ANALYSIS

The results shown if Fig. 5 resemble the typical CAT behavior: when a system parameter is changed the decay rate presents an irregular sequence of peaks on a smooth background. It is thus natural to look for a connection between CAT systems and our model. First of all, we notice that, due to the continuous nature of spectrum, the level dynamics of our system cannot be simply described in terms of avoided crossings. Nevertheless, we believe that the phenomenon underlying Fig. 5 retains much of the features of the CAT processes. In particular we think that the argument introduced in Ref. [9] to explain the tunneling irregularities of a quasi-integrable system, can be effectively used also in this model. In the cited reference, a connection between the peaks in the tunneling rate and the position of the nonlinear resonances in the classical phase space was found.

The analysis of Ref. [9] follows the line of Refs. [28,29] and is based on a semiclassical approximation that makes use of simple arguments. We shall assume that the classical system is only weakly perturbed by the external perturbation: the size of the chaotic region is considered to be small with respect to the portion of the classical phase space covered by regular trajectories so that the main effect of the perturbation is the appearance of chains of nonlinear island in the classical phase space as shown if Fig. 3. Under this condition, the area of the regular region, that is, the phase-space region encircled by the last unbroken torus inside of the stochastic layer, is much larger than $\hbar$. In this way, the regular region can accommodate several quantum states and the semiclassical approximation becomes meaningful.

As we discussed in the Introduction, the connection of classical nonlinear resonances to avoided crossings was early recognized [14–16]. For each nonlinear resonance in the classical motion we can find a level crossing in the quantum spectrum. These crossings are actually avoided but the modification of the levels can be so small as to become practically negligible for normal dynamics. In fact the level separation at the crossing point is related to the nonlinearity strength, which in our case is represented by the strength of the time dependent perturbation, and the former vanishes exponentially as the latter decreases [15,28,29]. Moreover, the levels modification decreases once the order of the nonlinear resonance increases. This can be seen as the counterpart of the KAM theorem because, strictly speaking, the level crossings are dense in the spectrum in the same way as nonlinear resonances are dense in the phase space, (see Sec. III), but only the lower order ones really affect dynamics.

This is still true if we consider a system with a spectrum similar to the one depicted in Fig. 1, but in this case we need to compare the smallness of the levels modification with another small effect, namely, with the level splitting due to the tunneling. It is clear that an avoided crossing, negligible if compared to the energy scale of system dynamics, can be, nevertheless, very important if we concentrate our attention on tunneling processes. The unperturbed splitting can be, in fact, significantly modified and the tunneling rate can change by order of magnitude.

The effect of tunneling modification due to isolated avoided crossings, i.e., to isolated nonlinear resonances, even if broadly discussed [13–19], received little attention in the CAT papers, which have been mainly concerned with the role of the stochastic layer and the classical transport therein as the main contribution to the barrier crossing. This effect is surely present, but its contribution is not always the most important, at least in a weak-chaos regime, where the effects of the nonlinear resonances can dominate the dynamics [9]. These prechaotic structures cannot, in fact, contribute to the classical transport over the barrier, because they are embedded in a regular region of unbroken tori, but they can perturb the quantum dynamics as discussed above.

This process, which we think to be deeply connected with CAT, in the weak-chaos regime can give a contribution to the tunneling rate modification even more important than the one connected to the chaotic region of the phase space. Moreover, due to the fact that the perturbing third state is not chaotic, it is possible to obtain a quantitative prediction of the avoided crossings and hence on the positions of the tunneling irregularities. We shall now review the derivation of this prediction.

Let us first note that the Hamiltonian under study is time dependent and that this prevents us from adopting the Einstein-Brillouin-Keller (EBK) quantization conditions [30] in their original form. However, following the work of Breuer and Holtaus [29], who, in turn, extended the method of the canonical operator as developed by Maslov and Fedoriuk [31], it is possible to adapt the EBK prescriptions to periodically time-dependent systems. The generalization of Ref. [29] is based on the prescription of Arnold [32] and leads to semiclassical quantization rules for the Floquet quasienergies and quasieigenstates [12]. This is made possible by a suitable extension of the phase space, including the time t as a coordinate and adding the corresponding conjugate momentum. In the one-dimensional case the semiclassical quantization prescriptions read

$$J=\frac{1}{2\pi}\oint_{\gamma_1} p\,dq=\hbar\left(n+\frac{\nu}{4}\right),$$

$$E_{n,m}=-\frac{1}{T}\int_{\gamma_2}(p\,dq-H\,dt)+\hbar\Omega m. \tag{7}$$

The meaning of the symbols adopted in this expression can be explained by remembering that in the one-dimensional case the extended space is the $\{q,p,t\}$ space. A regular trajectory is contained on a flux tube in this tridimensional space, flux tube that repeats itself periodically along the t direction. Thus J is the action associated with the quantized trajectory, γ_1 is a close path winding once around the flux tube and lying in the plane at a given time t and γ_2 is a path stretching itself out on the surface of the flux tube such that it can be continued adopting periodic boundary conditions. In other words, the path γ_2 in the extended tridimensional phase space, moves from an initial point lying on the plane $t=0$ to a final point lying on the plane $t=T$. Note also that we can choose to lay the path γ_1 on the Poincaré section of the flux tubes. Finally, if we restrict ourselves to considering the closed orbits inside the potential well, the Maslov index assumes only the value $\nu=2$.

Worth a detailed discussion is the structure of the quantized energy $E_{n,m}$ (for simplicity, we shall use the terms *energy* and *state* as equivalents of *quasienergy* and *quasistate*). While the index n, whose values are fixed by the first of Eqs. (7), has the usual role of principal quantum number, the index m, and the dependence of $E_{n,m}$ on this one, reflects the periodicity of the time dependent term of the Hamiltonian. In fact, an important aspect of the Floquet theory is the Brillouin zone structure of the energy spectrum: for each physical solution labeled by n we have an infinite series of representatives labeled by the value of m. Naturally, all of the physical information is contained in the first Brillouin zone $0\leqslant E_{n,m}<\hbar\Omega$, or equivalently, we can say that any solution of Eqs. (7) can be folded back to the first Brillouin zone by an appropriate choice of m.

At this point it is important to recall that the earlier formalism represents a valid quantization procedure only in closed systems, where the notion of energy levels is meaningful, and where the phase space is filled with regular tori. However, we are investigating the properties of states that lie well inside the stability region, and that present a small decay probability (see Fig. 5). In this condition, we believe that an analysis which, according to the prescriptions of CAT, connects the tunneling irregularities with the presence of avoided level crossings, can retain its validity. Let us proceed disregarding the continuity of the spectrum and looking for the presence of level crossings in the unperturbed Hamiltonian spectrum as a function of the forcing frequency Ω.

Following Ref. [29] we can look for the level crossings by replacing H with H_0, let us in fact recall that we are always considering a perturbative approach. The condition of crossing between states n and n' in the Brillouin zone yields the following equation:

$$H_0[\hbar(n+1/2)]+\hbar\Omega m=H_0[\hbar(n'+1/2)]+\hbar\Omega m'. \tag{8}$$

This equation can be simplified if we assume that $\hbar$ is so small as to make the quantity $\hbar(n-n')$ negligible. If we now expand the right-hand side of this equation around this small parameter, we obtain

$$\frac{dH_0}{dn}(n-n')+\hbar\Omega m=\hbar\Omega m', \tag{9}$$

which can be rewritten as

$$\frac{dH_0}{dJ}\frac{dJ}{dn}(n-n')=\hbar\Omega(m'-m) \tag{10}$$

or, by using $J=\hbar(n+1/2)$,

$$\omega_0=\frac{\Delta m}{\Delta n}\Omega+O(\hbar), \tag{11}$$

where $\omega_0\equiv\omega_0(J)=dH_0/dJ$ is the frequency of the unperturbed motion as a function of the classical action J, $\Delta n\equiv n'-n$ and $\Delta m\equiv m-m'$. The condition of levels crossing, in the limit of vanishing $\hbar$, can thus be obtained by solving the classical equation that corresponds to the condition for the onset of the classical nonlinear resonances.

Expression (11) confirms the result of the literature, namely, that for sufficiently small $\hbar's$, there is a correspondence between the presence of a nonlinear resonance in the classical phase space and a level crossing in the spectrum and, therefore, a correspondence between the tunneling irregularities and the nonlinear resonances. In other words, the tunneling peaks are the quantum manifestation of the resonant behavior of the underlying classical dynamics. The result of Eq. (11) is even more specific, in fact, it yields the following relation: the crossing of the two unperturbed levels $E^0_{n,m}$ and $E^0_{n',m'}$ is related to the superposition between the semiclassical quantization torus of one of the two states and the nonlinear resonance of appropriate order $\Delta m/\Delta n$.

This process admits a simple intuitive representation. Let us consider, for example, a quantum state located deeply in the well. Its quantization torus lies inside the well and the wave function in the semiclassical regime is mainly located around this torus. The decay rate in the unperturbed case can be obtained by the usual semiclassical calculation of the probability of barrier crossing. In a regime of weak chaos, if we turn the perturbation on, we have a high probability that nothing happens, due to the fact that most of the phase space inside the well remains unperturbed. But if we change an external parameter as, for example, the forcing frequency Ω, we obtain that the nonlinear resonances move in the phase space and, for particular values of the parameter, one of them can intersect the quantization torus, which is therefore destroyed. This would be probably reflected in a perturbation of the quantum state and thus in a modification of its decay rate. This is exactly what is described by Eq. (11).

As discussed before, Eq. (11) involves resonances of any order and this implies that a change of a system parameter indicates that a chosen level undergoes a virtually infinite number of crossings. However, the perturbation produced by the avoided crossings strongly depends on the order Δm of the resonance and in practice, if we restrict ourselves to the perturbative regime, the only significant crossings are those associated with the first-order resonances ($\Delta m=1$). On the other hand, the same happens in the classical dynamics, as discussed in Sec. III.

The simple prediction of Eq. (11) is valid in the strong semiclassical limit and is therefore unavailable for a numerical check, being the smallest value of $\hbar$ dictated by the computer limitations. The relation between nonlinear resonances and tunneling peaks has been indeed checked in Ref. [9] by means of a further expansion of Eq. (8). To do that we proceed by evaluating the second-order term in $\hbar(n-n')$. We obtain

$$\omega_0+\frac{1}{2}\hbar\frac{d\omega_0}{dJ}\Delta n=\frac{\Delta m}{\Delta n}\Omega+O(\hbar^2). \tag{12}$$

On the other hand,

$$\frac{d\omega_0}{dJ}\equiv\frac{d\omega_0}{dE}\frac{dE}{dJ}=\frac{d\omega_0}{dE}\omega_0. \tag{13}$$

Thus we can write Eq. (12) as follows:

$$\omega_0(E)=\frac{\Delta m}{\Delta n}\frac{\Omega}{\left(1+\frac{\hbar}{2}\frac{d\omega_0(E)}{dE}\Delta n\right)}+O(\hbar^2). \tag{14}$$

We give an analytical expression to $d\omega_0(E)/dE$, by using Eq. (5), as

$$\frac{d\omega_0}{dE}=\frac{\pi}{4k^2\sqrt{V_0}}\frac{1}{K(k)}\left(1-\frac{E(k)}{k'^2K(k)}\right), \tag{15}$$

where $k'^2=1-k^2$. Equation (14) represents a generalization of the classical nonlinear resonance condition: the frequency ratio is renormalized by means of a quantum correction proportional to $\hbar$.

By using this prediction we are now able to verify our conjecture about the validity of this method also in the present case. From Eq. (14) it is possible to predict the position of the peaks of decay. After fixing the values of Δm, Eq. (14) can be solved as a function of the energy E for several values of n. The graphical solution for the first-order crossings, $\Delta m=1$, is shown in Fig. 6, where the thick solid horizontal lines correspond to the semiclassical energies of the third and fourth eigenstates of the unperturbed Hamiltonian (3), i.e., of the states that we chose as initial condition in order to obtain the results of Sec. IV. The dotted and dashed curves represent the quantum renormalized energies of the classical nonlinear resonances of different order n (the order is indicated by the numbers in the figure). The crossings between the horizontal lines and the curves thus indicate the solutions of Eq. (14). Their position should also indicate the position of the peaks of the decay rate. This is, in fact, approximately true, as one can check by going back to Fig. 5, where we indicated the solutions showed in Fig. 6 with the numbered vertical dashed lines.

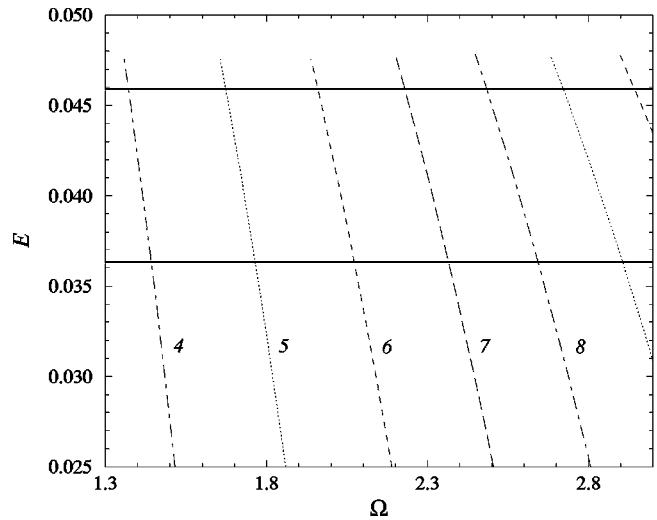

FIG. 6. The solutions of Eq. (14) represented by the crossings between the solid horizontal lines, indicating the energy of the third and fourth eigenstates, and the energy of various nonlinear resonances indicated by their order n (dashed and dotted lines). The crossings give the theoretical prediction on the position of the peaks of decay, which is reported in Fig. 5 as the vertical dashed lines. The values of the parameter are $V_0=0.048$, $\epsilon=0.005$, $\Omega=2$, $\hbar=0.025$. Arbitrary units.

We are now able to justify *a posteriori* the use of the semiclassical theory of this section in the present case. The main difference between the physical system of Ref. [9] and Hamiltonian (1) is the discreteness of energy levels. In the present work we cannot speak about level crossings and thus the calculations above could look invalidated. On the other hand, we showed that the avoided crossings are nothing more than the *trait d'union* between the nonlinear classical resonances and the peaks of the decay rate: the presence of the nonlinear resonance produces a level crossing that is reflected in a rate irregularity. The same happens for the system of Hamiltonian (1): the classical phase-space structures and the decay rate modifications are related even if the intermediate step is less clear due to the continuity of the quantum spectrum. Probably we could find a process similar to the avoided crossings, but we do not need to look for it as we showed that the quantum-classical connection works.

The role of the nonlinear resonances in the perturbation of tunneling seems thus to be established also in a system with a continuous spectrum, even if the prediction looks approximate. The nonperfect agreement between theory and numerical calculation can be traced back to two major approximations. The first is the finiteness of $\hbar$ that makes Eq. (14) slightly inaccurate, while the second and the most important would be the approximation related to consider the unperturbed states in Eq. (8). This last is in fact a double approximation, because it disregards the effect of the perturbation, but this is not so important as shown in Ref. [9], and the effect of the unperturbed decay that actually destroys the discrete levels picture we used. Nonetheless, we think that our results are not questionable and to make them clearer to the reader, we shall now resort to a graphical picture.

VI. A PHASE-SPACE REPRESENTATION: QUANTUM-CLASSICAL COMPARISON

As mentioned above, in order to connect the quantum dynamics to the classical phase-space characteristics, we must extend the concept of phase space to the quantum case. This can be done by using a phase-space representation of the quantum state and among all the different possibilities, we chose to use the Husimi representation, defined as

$$\rho(q,p)=\left\| \int_{-\infty}^{\infty} dx\,\alpha_{q,p}(x)\,\psi(x)\right\|^2, \tag{16}$$

where $|\alpha_{q,p}\rangle$ is a minimum indetermination state (coherent state) centered in (q,p). Using Eq. (16) we can obtain a phase representation of a quantum state in terms of the positive definite distribution $\rho(q,p)$ [33].

In particular, we can calculate the Husimi distribution of the initial state and, due to the choice of quasistationary initial conditions, this will allow us to obtain the correspondence we are looking for in an easy way. In fact, for small decay probability, we can safely disregard the dynamics inside the well for the times we explored. This means that the phase-space representation of the quantum state practically does not change during the time interval considered in Fig. 5, and that in the comparison between quantum and classical phase space we can limit ourselves to dealing with the initial quantum distribution.

In Fig. 7 we show portraits of the classical phase space for increasing values of Ω. For clarity, we draw only the stochastic web and the nonlinear resonances island structures, all the rest of the phase space being filled with regular tori. As explained in Sec. III, due to the choice of the particular form of the time dependent perturbation, to the first-order in ϵ we have only resonances of the form $\omega_0(E)=\Omega/n$ where $\omega_0(E)$ is the frequency of the motion inside the unperturbed well. The frequency $\omega_0(E)$ is a decreasing function of the energy E for $0<E<2V_0$, being equal to $\sqrt{4V_0}$ for $E=0$ and vanishing for $E=2V_0$ that corresponds to the separatrix motion, see Eqs. (5). This means that as we approach the separatrix we find resonances of larger-order n. We realize that as the value of Ω is increased, the nonlinear resonances move inside the phase space, getting closer to the center of the well and eventually disappearing when the relation (4) cannot be fulfilled any more. In the meanwhile, new resonances appear, moving out of the stochastic layer that is the region of the overlapping of all the infinite resonances of higher-order n.

In this motion toward the center of the well, the various resonances cross the region of the phase space that is occupied by the Husimi distribution of the initial quantum state [shaded area in Fig. 7(b)] and, as discussed in Sec. V, we expect this to be related to the peaks of Fig. 5. The region of parameters explored in Figs. 5 and 7 is the same and from a first inspection we actually realize that the number of resonances crossing the shaded area corresponds to the number of peaks in the decay rate. This first result is already convincing but we can be more precise by singling out the phase-space portraits that correspond to the decay peaks.

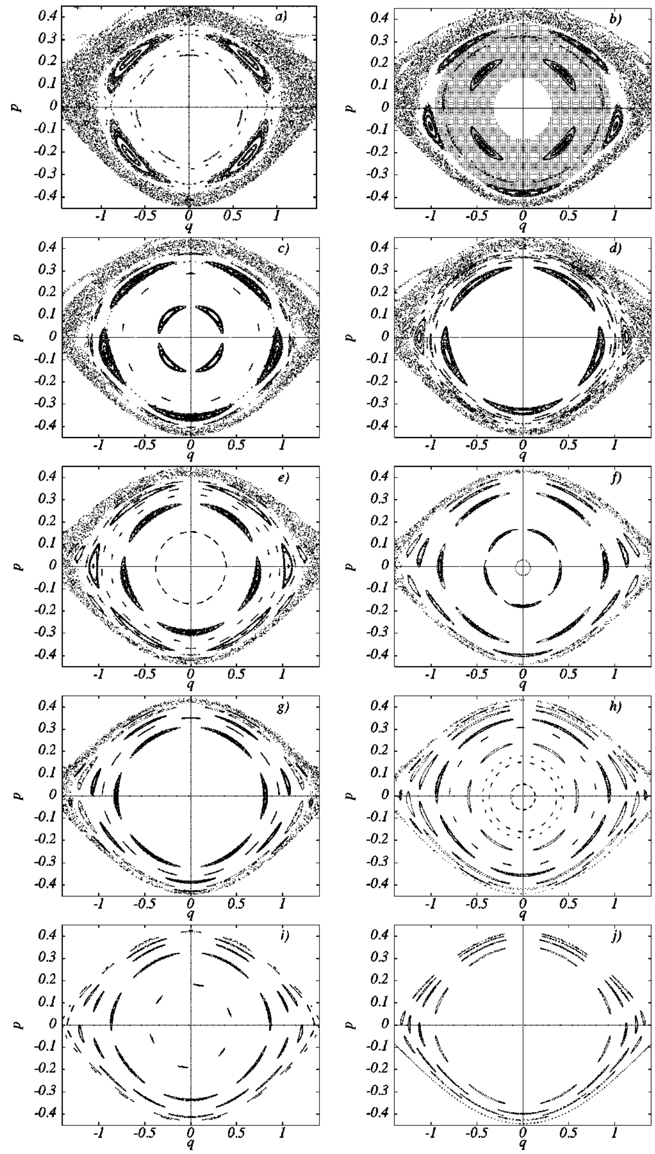

FIG. 7. Classical phase-space portraits for different values of the driving frequency Ω. The values of the parameter are $V_0=0.048$, $\epsilon=0.005$, $\hbar=0.025$. The value of Ω, increasing from (a) to (j), is $\Omega=1.4$, 1.6, 1.7, 1.8, 1.9, 2.1, 2.2, 2.4, 2.5, 2.7. The shaded area in (b) indicates the area under the Husimi distribution of the initial state, which in this case is the fourth eigenstate of the unperturbed well. Arbitrary units.

This is done in Fig. 8 where we show the classical phase-space structures and we indicate by the two continuous lines the borders of the Husimi distribution of the initial quantum state. The expected results are confirmed by the observation that the peaks in the quantum decay rate are related to the modification of the classical phase space under the quantum initial distribution by a nonlinear resonance. The perturbation seems to be effective when a nonlinear resonance enters the external border of the initial state distribution rather than when the nonlinear resonance passes over the center of the distribution where the quantization torus is located. This is qualitatively in agreement with Eq. (14) that predicts that the perturbation appears when the nonlinear resonance is close to

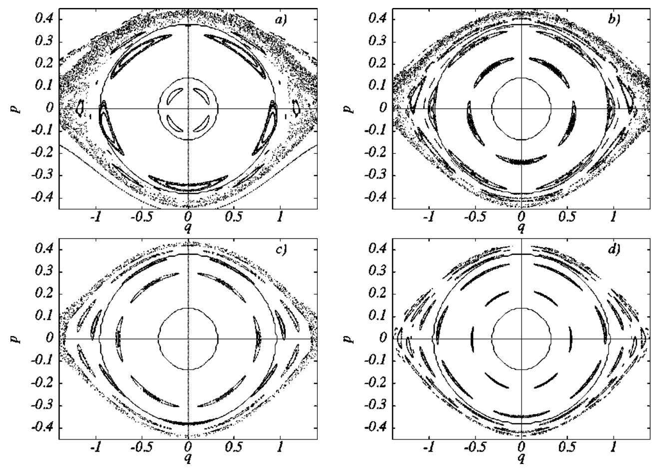

FIG. 8. Classical phase-space portraits for different values of the driving frequency Ω. The values of the parameter are $V_0=0.048$, $\epsilon=0.005$, $\hbar=0.025$. The values of Ω are chosen as the values corresponding to the peaks in Fig. 5(a), i.e., $\Omega=1.73$ (a), 2.015 (b), 2.25 (c), and 2.44 (d). The two continuous circular curves indicate the outer and inner borders of the Husimi distribution of the initial quantum state (the fourth eigenstate). The border is defined arbitrarily as the contour level of the distribution at 0.2 times the maximum height. Arbitrary units.

the semiclassical quantization torus, how much closer being dictated by the quantum correction that modifies Eq. (14) with respect to Eq. (4). In fact, the quantum correction to the forcing frequency Ω is positive (the term $d\omega_0(E)/dE$ is negative for energy smaller than the barrier heights and thus the renormalized frequency is always larger than Ω) and this means that the decay peaks should appear for values of Ω smaller than the values for which the nonlinear resonances superpose the quantization torus. This is precisely what happens: the peaks correspond to the approaching of the nonlinear resonances from the outside.

This result comes along with a simple intuitive explanation: when a nonlinear resonance enters the external border of the initial state, part of the initial distribution is moved outward by the islands structure, this produces a higher probability of tunneling across the barrier and thus an increase of the decay. This effect becomes less important once the nonlinear resonance penetrates inside the shaded region, until it disappears when the resonance is completely embedded in the central part of the distribution.

The results of Sec. V have been thus confirmed by the comparison of this section, making clearer the role of the classical nonlinear resonances.

VII. CHAOS-ASSISTED DECAY VERSUS CHAOS-ASSISTED TUNNELING

When we reviewed the numerical results we noticed that we found only enhancement of the decay compared to the unperturbed condition. This is not the case in CAT, where the avoided crossings can produce both enhancement and decrease of tunneling, which can also vanish for particular values of the parameters [3]. For this reason we wrote in the Introduction that the term *assisted* used in CAT is not really appropriate, but it seems that it could be better used in the present context. To explain this different behavior we conjecture that this can be seen as a consequence of the continuity of the spectrum in our system. In fact, the quenching of the tunneling is produced by the accidental degeneracy of the levels of the tunneling doublet, as seen in Fig. 1. This degeneracy is due to the modification induced by the crossing with the third level. In our case we do not have discrete levels, but a continuous density of states and thus the former picture simply does not apply. In other words, for every value of the parameter the modification of the spectrum due to the presence of avoided crossings cannot lead to a complete degeneracy of the states and thus to a complete quenching of the decay. A similar situation and a graphical representation of this process can be found in a recent paper [34]. On the other hand, this continuous spectrum characteristic cannot completely rule out the possibility that nonlinear resonances could produce also a decrease of the unperturbed decay rate, which could be present in some region of parameters or for different choices of initial conditions.

VIII. CONCLUSIONS

A numerical calculation of the decay from a potential well due to tunneling showed that, in presence of classical chaos, the decay can be strongly enhanced and that this enhancement depends on the system parameters in a resonantlike way. A qualitative inspection of the classical phase-space structure revealed a connection between the peaks in the decay probability and the presence of classical nonlinear resonances in the region of phase space occupied by the Husimi distribution of the initial state. This correspondence has been quantitatively explained using a semiclassical result that has been shown to be valid in the case of chaos-assisted tunneling.

This semiclassical prediction is based on a picture that singles out the classical nonlinear resonances as the main factor responsible for the fluctuations of the tunneling rate. The presence of a nonlinear resonance in the region of the phase space that represents the support of the quantum state leads to the tunneling perturbation, in a way that appears as the quantum manifestation of the classical ‘‘small denominators’’ problem [21]. This is a direct connection between the modification of a purely quantum effect, the tunneling, and the classical phenomenon of destruction of integrable dynamics that is at the basis of the chaotic behavior.

Finally, by studying a decay process, we addressed an unbounded system, with a continuous spectrum, and this allowed us to point out the differences between this process of nonlinearly assisted decay and CAT.

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